radar range and position-velocity parameters as

$$\frac{d}{dt}\frac{(R^2)}{2} - \frac{\dot{r}}{r}\frac{R^2}{2} = \frac{\dot{r}}{r}\frac{(r^2 - r_e^2)}{2} + r\, r\dot{\sigma}\cos\mu \sin\sigma \quad (10)$$

The integrating factor for Eq (10) is $\exp[-\ln(r/r_0)] = (r_0/r)$

After multiplying both sides of Eq (10) by the integrating factor, integrating between initial and final values, and solving for R^2, there results

$$\frac{R^2}{2} = \frac{R_0^2}{2}\frac{r}{r_0} + \frac{r_0 - r}{2}\left(\frac{r_e^2}{r_0} - r\right) + r r_e \cos\mu(\cos\sigma_0 - \cos\sigma) \quad (11)$$

where the subscript zero indicates the initial conditions of the integration interval Equation (11) is the desired range equation in its general form based upon a nonrotating earth in order to maintain the parameter $\cos\mu$ equal to a constant The form of Eq (11) is applicable to any analytically definable trajectory that is confined to a prescribed great circle plane Examples are satellite and ballistic missile trajectories that are definable by the Keplerian equations during the target's passage through the radar surveillance volume, constant flight path aircraft trajectories in which r = const and $\sigma = Vt/r$, earth-moon trajectories, and guided missile trajectories with pitch control only The only requirement for its application is that the earth-centered radius r of the trajectory be known as a function of time or σ

Elevation angle tracking equation

Using the results of the radar range tracking equation, a simple analysis leads to the elevation angle tracking equation Referring to Fig 2, let

X_g = radar ground range to the target
Z = radar zenith component of the radar range

Then

$$X_g = R\cos\phi$$
$$Z = R\sin\phi \quad (12)$$
$$X_g^2 + (r_e + Z)^2 = r^2$$

Upon taking the total differential of Eq (12) and dividing each term by the product $(r_eR)dt$, the following first-order differential equation results:

$$\frac{d(\sin\phi)}{dt} + \frac{(dR/dt)}{R}\sin\phi = \frac{r(dr/dt)}{r_eR} - \frac{(dR/dt)}{r_e} \quad (13)$$

The integrating factor for Eq (13) is $\exp[\ln(R/R_0)] = R/R_0$

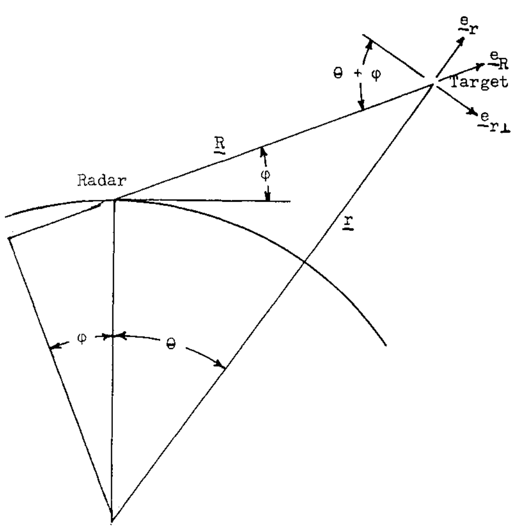

Fig 2 Earth-centered geometry in the plane of the radar site and target positions

Thus, integrating Eq (13) between the initial and final conditions, a form of the elevation angle updating equation is

$$\sin\phi = \frac{R_0}{R}\sin\phi_0 + \frac{r^2 - r_0^2}{2r\,R} + \frac{R_0^2 - R^2}{2r\,R} \quad (14)$$

An equivalent form of Eq (14) can be derived from the complete h (height) to Z conversion:

$$Z = h + (h^2 - R^2)/2r \quad (15)$$

Multiplying Eq (15) through by $2r_e$ and completing the square results in

$$\sin\phi = (r^2 - r_e^2 - R^2)/2r_eR \quad (16)$$

Equation (16) is equivalent to Eq (14)

Azimuth tracking equation

Referring to the geometry of Fig 1, let the radar Cartesian coordinate system be defined by the X, Y, and Z axes in which

Z = zenith axis
Y = axis that lies in the β,γ plane directed toward the target plane
X = axis that is parallel to the target plane defined by a right-handed coordinate system

In addition, let

A = azimuth angle of the radar range vector relative to the Y axis

Then, from spherical trigonometric relationships, it can be shown that

$$\sin A = \sin\sigma/\sin\theta = r\sin\sigma/R\cos\phi$$

In terms of the direction cosine angle ξ relative to the Y axis, $\sin\xi = r\sin\sigma/R$ from the relationship $\sin\xi = \sin A\cos\phi$

Reference

[1] Kumagai, T T, "Perturbational variations in a ballistic missile or satellite orbit about an oblate earth," AIAA J 1, 419–423 (1963)

Density Distribution over a Moving Circular Plate in Free-Molecule Flow (*U*)

K J Touryan*
Sandia Laboratory, Albuquerque, N Mex

The free-molecule flow field is calculated on the front side of a moving circular plate including the effect of finite area distribution on the density gradient of the reflected molecules Calculations are carried out both for diffuse and specular reflections, and the results are compared with those obtained from differential area considerations

Bird[1] made a thorough analysis of the free-molecule flow field around moving bodies of different geometries with surface elements located at the origin To study the effect of finite distribution of surface elements on this flow field on the plate axis as well as on the other arbitrary locations, we start with Bird's analysis

Received September 23, 1963
* Staff Member, Aerophysics Section

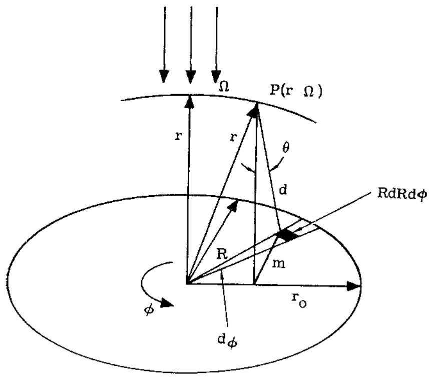

Fig 1 Geometric relationships for diffuse reflection

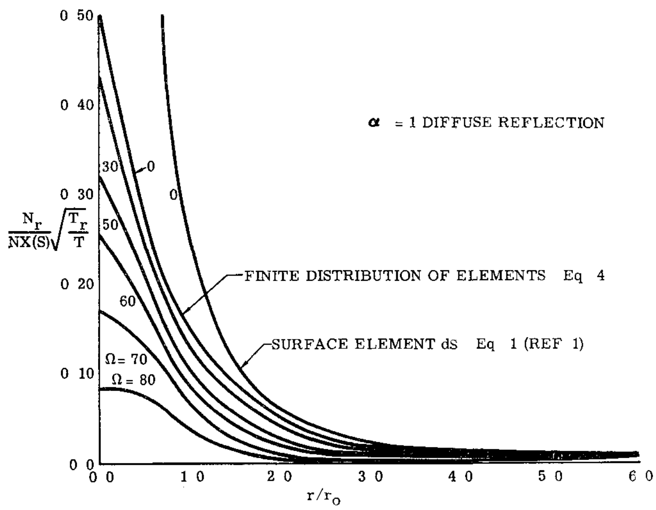

Fig 2 Density gradients on circular plate of radius r_0 at normal incidence

A Diffuse Reflection ($\alpha' = 1$)

The expression for the reflected molecules can be written as

$$dN_r = \frac{Nx(s)\cos\Omega}{4\pi d^2}\left(\frac{T}{T}\right)^{1/2} dS$$

or

$$N_r = \frac{Nx(s)}{4}\cos\Omega\left(\frac{r_0}{r}\right)^2 \text{ for } r_0 \ll r,\ dS = \pi r_0^2 \qquad (1)$$

where

N = number of molecules per unit volume
T = temperature
Ω = angle between surface normal and molecular velocity vector
$x(s)$ = $\exp(-s^2\sin^2\theta) + \pi^{1/2}s\sin\theta[1 + \text{erf}(s\sin\theta)]$
s = speed ratio $U/(2RT)^{1/2}$
S = surface
θ = angle of incidence of surface to stream
d = defined in Fig 1
r = reflected molecules (subscript)

From Fig 1, the following geometric relationships hold:

$$dS = RdRd\phi\cos\theta' \qquad (1a)$$

$$\cos\theta' = (r\cos\Omega)/d \qquad (1b)$$

$$d = (m^2 + r^2\cos^2\Omega)^{1/2} \qquad (1c)$$

$$m^2 = R^2 + r^2\sin^2\Omega - 2rR\sin\Omega\cos\phi \qquad (1d)$$

Substitution of these into Eq (1) gives

$$dN = \frac{Nx(s)\,(T/T_r)^{1/2}}{4\pi}\,\frac{r\cos^2\Omega RdRd\phi}{(R^2 - 2rR\sin\Omega\cos\phi + r^2)^{3/2}} \qquad (2)$$

In the special case of $\Omega = 0$, Eq (2) integrates in the closed form to give[1]

$$N = \frac{Nx(s)}{2}\left(\frac{T}{T}\right)^{1/2}\left\{1 - \frac{1}{[(r_0/r)^2 + 1]^{1/2}}\right\} \qquad (3)$$

In general, one has, after integrating over R, $0 \le R \le r_0$,

$$N = \frac{Nx(s)(T/T_r)^{1/2}}{4\pi}\cos^2\Omega \times \int_0^{2\pi}\left\{1 - \frac{1}{[(r_0/r)^2 - 2(r_0/r)\sin\Omega\sin\phi + 1]^{1/2}} + \frac{\sin^2\Omega\cos^2\phi}{1 - \sin^2\Omega\sin^2\phi} + \frac{\sin\Omega\cos\phi(r_0/r - \sin\Omega\cos\phi)}{(1 - \sin^2\Omega\cos^2\phi)[(r_0/r)^2 - 2(r_0/r)\sin\Omega\sin\phi + 1]^{1/2}}\right\} d\phi \qquad (4)$$

These integrals are all elliptic in form For example, the second term under the integral sign can be represented by

$$\frac{2}{(a + b)^{1/2}} K\left(\frac{2b}{a + b}\right)^{1/2}$$

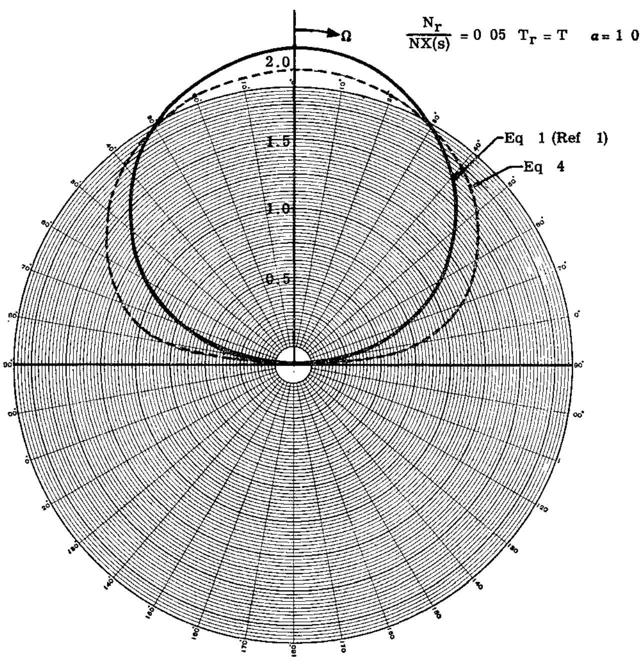

Fig 3 Contours of constant number density in the cloud of diffusely reflected molecules from a circular plate of radius R_0

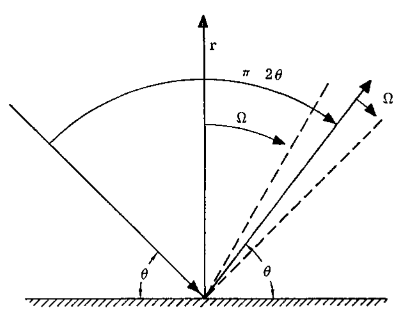

Fig 4 Geometric relationships for specular reflection

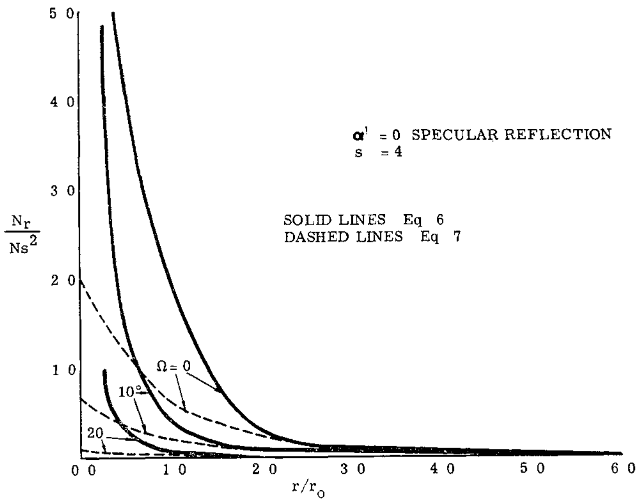

Fig 5 Contours of constant density in the cloud of specularly reflected molecules from a circular plate of radius r_0

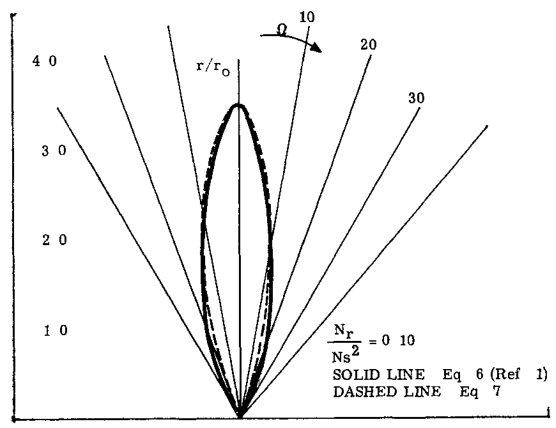

Fig 6 Contours of constant number density in the cloud of specularly reflected molecules from a circular plate of radius r_0

where $a = (r_0/r)^2 + 1$, $b = 2(r_0/r)\sin\Omega$, and K = elliptic integral of the first kind The results of a numerical integration of these integrals are plotted in Fig 2 with

$$\frac{N_r}{Nx(s)}\left(\frac{T_r}{T}\right)^{1/2} \text{ vs } \frac{r}{r_0}$$

for different values of Ω Figure 3 represents the constant density contours in the cloud of diffusely reflected molecules from the plate surface given by both Eqs (1) and (4)

B Speculai Reflection ($\alpha' = 0$)

The modified expression for dN now becomes[1]

$$dN = \frac{N \sin\theta s^2 \exp(-s^2 \tan^2\Omega')dS}{(1-\alpha')^{1/2}\pi r^2 \cos^3\Omega'} \tag{5}$$

where α' = accommodation coefficient, and Ω' = angle between number density of reflected molecules and the reflected stream direction (Fig 4) Now $\sin\theta = 1$ for the plate set normal to the stream Then

$$dN_r = \frac{Ns^2 \exp(-s^2 \tan^2\Omega')dS}{(1-\alpha')^{1/2}(\pi r^2)\cos^3\Omega'} \tag{6}$$

From Sec A,

$$dS = RdRd\phi \frac{r\cos\Omega}{(R^2 - 2rR\sin\Omega\cos\phi + r^2)^{1/2}}$$

Then

$$N_r = \frac{Ns^2\exp(-s^2\tan^2\Omega)}{\pi(1-\alpha')^{1/2}\cos^2\Omega} \times$$

$$\int_0^{2\pi}\left\{1 - \frac{1}{[(r_0/r)^2 - 2(r_0/r)\sin\Omega\cos\phi + 1]^{1/2}} + \frac{\sin^2\Omega\cos^2\phi}{1-\sin^2\Omega\cos^2\phi} + \frac{\sin\Omega\cos\phi(r_0/r - \sin\Omega\cos\phi)]}{(1-\sin^2\Omega\cos^2\phi)(\sin\Omega\cos\phi + 1)^{1/2}}\right\}d\phi \tag{7}$$

The results are plotted in Figs 5 and 6 where curves obtained from Eq (6), $dS = \pi r_0^2$, are compared to the curves obtained from the more exact equation (7) for $s = 4$ Note that the effect of the finite distribution of elements is much less pronounced for the specular case than for the diffuse reflection

Reference

[1] Bird, G A , ' The free molecule flow field of a moving body in the upper atmosphere, ' *Rarefied Gas Dynamics*, edited by L Talbot (Pergamon Press, Inc , New York, 1960), pp 245–260

Trajectories with Constant Normal Force Starting from a Circular Orbit

Richard R Auelmann*
Philco Corporation, Newport Beach, Calif

The totality of motions for a particle initially in a circular Kepler orbit and acted upon by a constant, in-plane normal force is determined The orbits lie in a ring bounded by two circles, the first with radius equal to the radius of the initial Kepler orbit and the second with radius dependent on the normal force The second circle lies outside the first circle when the normal force is outward and lies inside when the normal force is inward The radius of the second circle cannot exceed twice the radius of the first circle and is reached only when the normal force is 0 230 times the gravity force at the initial radius The point of central attraction is reached only when the normal force is 2 809 times the gravity force at the initial radius The orbit path oscillates periodically between the two circles However, the orbits are not, in general, periodic since they do not close When the magnitude of the normal force is small, the orbits are direct, whereas when the force is large, the orbits are diiect near the first circle and retrograde near the second circle

Introduction

A PARTICLE is moving in a circular Kepler orbit when a constant force is applied perpendicular to the instantaneous velocity vector and in the plane of motion What is the resulting motion of the particle?

This problem with the applied force in the normal direction and three other closely related problems with the applied

Received October 28, 1963 This paper is based on work performed for NASA, Contract No NAS 8-5248

* Senior Staff Member, Aeronutronic Division Member AIAA